Available online at www.sciencedirect.com
SCIENCE DIRECT®

JOURNAL OF GEOMETRY AND PHYSICS

Journal of Geometry and Physics 52 (2004) 353–363

www.elsevier.com/locate/jgp

Area monotonicity for spacelike surfaces with constant mean curvature

Rafael López*

Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, Spain

Received 25 March 2003; received in revised form 24 March 2004; accepted 24 March 2004
Available online 30 April 2004

Abstract

We study compact spacelike surfaces with constant mean curvature in the three-dimensional Lorentz–Minkowski space $\mathbb{L}^3$. When the boundary of the surface is a planar curve, we obtain an estimate for the height of the surface measured from the plane Π that contains the boundary. We show that this height cannot extend more that $A|H|/(2\pi)$ above Π, where A and H denote, respectively, the area of the surface that lies over Π and the mean curvature of the surface. Moreover, this estimate is attained if and only if the surface is a planar domain (with $H = 0$) or a hyperbolic cap (with $H \neq 0$).

MSC: 53C42; 53B30

JGP SC: Differential geometry

Keywords: Spacelike surface; Constant mean curvature; Co-area formula

1. Introduction and statement of results

In differential geometry, monotonicity formulae refer to control the variation of some of kind of energy (e.g. area, volume, etc.) in terms of the local geometry of the manifold. This can easily see in the theory of minimal and constant mean curvature surfaces [8,13]. In general relativity, the measure of matter is in a given region of a spacetime leads to the study of the monotonicity of the Hawking quasi-local mass of a connected surface under inverse mean curvature flow. For further information on this physical interpretation, see the review article of Bray [5].

* Tel.: +34-958-24-04-72; fax: +34-958-24-32-81.
E-mail address: rcamino@ugr.es (R. López).

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2004.03.008

Recall that the Lorentz–Minkowski space $\mathbb{L}^3$ is the space $\mathbb{R}^3$ endowed with the metric $\langle\, ,\, \rangle = (\mathrm{d}x_1)^2 + (\mathrm{d}x_2)^2 - (\mathrm{d}x_3)^2$, where $x = (x_1, x_2, x_3)$ are the canonical coordinates in $\mathbb{R}^3$. An immersion $x : \Sigma \to \mathbb{L}^3$ of a smooth surface Σ is called spacelike if the induced metric on the surface is positive definite. In this setting, the notions of the first and second fundamental form, and the mean curvature are defined in the same way as on a surface in Euclidean space. This article deals with spacelike immersed surfaces $x : \Sigma \to \mathbb{L}^3$ with *constant mean curvature* H. It is well known that such surfaces are critical points of the area functional for variations which preserve a suitable volume function. On the other hand, in relativity there is interest of finding real-valued functions on a given spacetime, all of whose level sets have constant mean curvature. The mean curvature function may then be used as a global time coordinate and provide a time gauge which is important in the study of singularities, the positivity of mass and gravitational radiations (see e.g. [7,12]).

In this paper, we shall obtain an area monotonicity formula that gives a dependence of the area of a compact spacelike surface with constant mean curvature in $\mathbb{L}^3$ with respect to the height of the surface. Exactly, we show the following theorem.

Theorem 1. *Let $x : \Sigma \to \mathbb{L}^3$ be a spacelike immersion of a compact surface with boundary included in a plane Π. Assume the mean curvature H of the immersion is constant. If h denotes the height of Σ with respect to Π, we have*

$$h \le \frac{|H|A}{2\pi}, \tag{1}$$

where A is the area of the region of Σ above the plane Π. The equality holds if and only if Σ is a planar domain ($H = 0$) or a hyperbolic cap ($H \neq 0$).

On the other hand, hyperbolic caps show that if we fix the mean curvature H, there exist spacelike graphs with constant mean curvature H and with *arbitrary* height. This do not occurs in Euclidean setting.

However, as far as I know, the only similar estimate as (1) was obtained by Bartnik and Simon for graphs (see [4, Eq. (2.15)]). The monotonicity that they obtain measures the variation of the area of the surface that lies in the domain $S_r(p)$ defined by

$$S_r(p) = \{x \in \mathbb{L}^3; \langle x - p, x - p\rangle < r^2\}.$$

They prove that if the boundary of Σ satisfies $\partial\Sigma \cap S_r(p) = \emptyset$, the following limit holds

$$\lim_{r\to 0} \frac{\mathrm{area}(\Sigma \cap S_r(p))}{r^2} = \pi.$$

Our approach is based in the Federer co-area formula for the height function, a formula that measures the flux of the surface across the boundary, jointly the classical isoperimetric inequality in the plane.

This paper consists of four sections. Section 2 is a preparatory section, where we will give some definitions and notations, and we will mention basic properties of the compact spacelike surfaces with constant mean curvature. Section 3 will be devoted to the tangency principle. In this section, we will derive an estimate of the height of a graph with constant mean curvature in terms of the diameter of the domain. Last, Theorem 1 will be proved in Section 4.

2. Notations and preliminaries

Throughout this section as well as the following one, we shall consider arbitrary dimension. First we will mention basic fact about the compact spacelike hypersurfaces with constant mean curvature and we will state a flux formula for this kind of hypersurfaces, which we need for the proof of our main result. Let $\mathbb{L}^{n+1}$ denote the $(n+1)$-dimensional Lorentz–Minkowski space, that is, the real vector space $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric:

$$\langle\, ,\, \rangle = (\mathrm{d}x_1)^2 + \cdots + (\mathrm{d}x_n)^2 - (\mathrm{d}x_{n+1}^2),$$

where $(x_1, \ldots , x_{n+1})$ are the canonical coordinates in $\mathbb{R}^{n+1}$. In what follows, a smooth immersion $x : \Sigma \to \mathbb{L}^{n+1}$ of an n-dimensional connected manifold Σ is said to be a *spacelike hypersurface* if the induced metric via x is a Riemannian metric on Σ, which, as usual, is also denoted by $\langle\, ,\, \rangle$. If $\mathbf{e}_{n+1} = (0, \ldots , 0, 1)$ and since any unit vector field N of Σ is timelike, the product $\langle N, \mathbf{e}_{n+1}\rangle$ cannot vanish anywhere. We shall choose the orientation of our hypersurfaces so that $\langle N, \mathbf{e}_{n+1}\rangle < 0$, that is, N will point upwards. We say then that N is future-directed.

Let us assume that Σ is a compact spacelike hypersurface. Since Σ cannot be closed in $\mathbb{L}^{n+1}$, the hypersurface Σ has non-empty boundary $\partial\Sigma$. If Γ is an $(n-1)$-dimensional closed submanifold in $\mathbb{L}^{n+1}$, we say that $x : \Sigma \to \mathbb{L}^{n+1}$ is a hypersurface with boundary Γ if the restriction $x : \partial\Sigma \to \Gamma$ is a diffeomorphism.

Let now ∇^0 (resp. ∇) denote the Levi–Civita connection of $\mathbb{L}^{n+1}$ (resp. Σ). The Gauss and Weingarten formulas for Σ in $\mathbb{L}^{n+1}$ are, respectively

$$\nabla^0_X Y = \nabla_X Y - \langle AX, Y\rangle N$$

and

$$A(X) = -\nabla^0_X N$$

for any tangent vectors fields $X, Y \in \mathcal{X}(\Sigma)$ and A stands for the shape operator associated to N. Then the second fundamental form σ and the mean curvature H of Σ are defined by

$$\sigma(X, Y) = -\langle\nabla^0_X Y, N\rangle, \tag{2}$$

$$H = \frac{1}{n}\,\mathrm{trace}\,\sigma = -\frac{1}{n}\mathrm{trace}(A) = -\frac{1}{n}\sum_{i=1}^{n}\langle\sigma(\mathbf{v}_i, \mathbf{v}_i), N\rangle, \tag{3}$$

where $X, Y \in \mathcal{X}(\Sigma)$ and $\{\mathbf{v}_i; 1 \le i \le n\}$ is a smooth tangent frame along Σ. We call Σ a hypersurface with constant mean curvature if the function H is constant in Σ.

Locally a spacelike hypersurface Σ is given as the graph of a function $u = u(x_1, \ldots , x_n) : \Omega \to \mathbb{R}$, Ω a domain of $\mathbb{R}^n$, with the condition $|Du|^2 < 1$ that means precisely that the graph defined by u is spacelike. The future-directed orientation is

$$N = \frac{(Du, 1)}{\sqrt{1 - |Du|^2}}, \qquad \langle N, \mathbf{e}_{n+1}\rangle < 0. \tag{4}$$

The first and second fundamental forms of Σ are then given, respectively, by

$$g_{ij} = \delta_{ij} - u_i u_j, \qquad A_{ij} = \frac{u_{ij}}{\sqrt{1-|Du|^2}},$$

where $u_i = D_i u \equiv \partial u/\partial x_i$ and $u_{ij} = D_i D_j u$. For graphs, the mean curvature H of the graph of u is expressed by

$$(1-|Du|^2)\sum_{i=1}^{n} u_{ii} + \sum_{i,j=1}^{n} u_i u_j u_{ij} = nH(1-|Du|^2)^{3/2}. \tag{5}$$

This equation can alternatively be written in divergence form:

$$\operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = nH. \tag{6}$$

In the study of spacelike hypersurfaces with constant mean curvature we state a result which will be needed in the proof of the main result (see [1,2]).

Lemma 2 (Flux formula). *Let $x : \Sigma \to \mathbb{L}^{n+1}$ be a spacelike immersion of a compact hypersurface with boundary $\partial\Sigma$. Denote by ν the inward pointing unit conormal vector along $\partial\Sigma$. If the mean curvature H is constant, then for any fixed vector $a \in \mathbb{L}^{n+1}$ we have*

$$H\int_{\partial\Sigma} \det(x, \tau_1, \ldots, \tau_{n-1}, a)\,\mathrm{d}s + \int_{\partial\Sigma} \langle \nu, a\rangle\,\mathrm{d}s = 0, \tag{7}$$

where $\{\tau_1, \ldots, \tau_{n-1}\}$ is a unit tangent frame to $\partial\Sigma$. If $x(\partial\Sigma)$ is included in a hyperplane $\Pi = a^{\perp}$, with $\langle a, a\rangle = -1$, then we obtain the flux formula:

$$\int_{\partial\Sigma} \langle \nu, a\rangle\,\mathrm{d}s = -nH\,\mathrm{vol}(\partial\Sigma), \tag{8}$$

where $\mathrm{vol}(\partial\Sigma)$ *is the algebraic volume of $\partial\Sigma$.*

Remark 3. The so-called flux formula can be viewed as the physical equilibrium between the forces of the surface tension of Σ that act along its boundary with the pressure forces that act on the bounded domain by $\partial\Sigma$. More generally, if we cut Σ in a collection of opens, then the surface tension along the cuts and pressure through the caps must balance.

The rest of the section will be devoted to study the hyperbolic hyperplanes and their role in the Lorentzian setting. We now review the construction of such hypersurfaces. After an isometry of $\mathbb{L}^{n+1}$, hyperbolic hyperplanes are defined as follows: for each $\rho > 0$ and $p \in \mathbb{L}^{n+1}$, let

$$\mathcal{H}_\rho(p) = \{x \in \mathbb{L}^{n+1};\ \langle x-p, x-p\rangle = -\rho^2\}.$$

Each one of such hypersurfaces has exactly two components,

$$\begin{aligned}\mathcal{H}_\rho(p) &= \mathcal{H}^+_\rho(p) \cup \mathcal{H}^-_\rho(p)\\ &= \{x \in \mathcal{H}_\rho(p);\ \langle x-p, \mathbf{e}_{n+1}\rangle < 0\} \cup \{x \in \mathcal{H}_\rho(p);\ \langle x-p, \mathbf{e}_{n+1}\rangle > 0\}.\end{aligned}$$

With the future-directed orientation, the mean curvature of $\mathcal{H}^+_\rho(p)$ and $\mathcal{H}^-_\rho(p)$ is $1/\rho$ and $-1/\rho$, respectively.

We call *hyperbolic caps* the compact pieces of hyperbolic hyperplanes whose boundary is a $(n-1)$-sphere, that is, the compact hypersurfaces that stem from the intersection of $\mathcal{H}^+_\rho(p)$ and $\mathcal{H}^-_\rho(p)$ with horizontal hyperplanes. Exactly, for each $p \in \mathbb{L}^{n+1}$ and $r > 0$, consider the disc $D_r(p) = \{(x_1, \ldots, x_n, 0) \in \mathbb{L}^{n+1}; (x_1 - p_1)^2 + \cdots + (x_n - p_n)^2 < r^2\}$. Define the upper hyperbolic cap $\mathcal{H}^+_\rho(p, r)$ (resp. the lower hyperbolic cap $\mathcal{H}^-_\rho(p, r)$) as the piece of $\mathcal{H}^+_\rho(p)$ (resp. $\mathcal{H}^-_\rho(p)$) that lies in the cylinder $\{(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}; (x_1 - p_1)^2 + \cdots + (x_n - p_n)^2 < r^2, x_{n+1} \in \mathbb{R}\}$. Hyperbolic caps $\mathcal{H}^\pm_\rho(p, r)$ are the graphs of the functions $u^\pm_\rho$ defined in $D_r(p)$ given by

$$u^\pm_\rho(x_1, \ldots, x_n) = p_{n+1} \pm \sqrt{\rho^2 + \sum_{i=1}^n (x_i - p_i)^2},$$

with the boundary condition:

$$u^\pm_{\rho|\partial D_r(p)} = p_{n+1} \pm \sqrt{r^2 + \rho^2}.$$

Both functions, and the corresponding graphs, are useful as barrier hypersurfaces in establishing boundary height and gradient estimates. In fact, the steepness of such hyperbolic caps at a given height is an upper bound for the steepness of any of a comparison constant mean curvature graph, at corresponding heights. These C^0- and C^1-estimates have as immediate application, the solvability of the Dirichlet problem when Ω is a convex domain. More general, Bartnik and Simon [4] proved existence and regularity for hypersurfaces with prescribed mean curvature and boundary values φ provided the function φ bounds some spacelike surface. The reader can see the techniques in [3,14].

3. The tangency principle and consequences

In this section, we will state the tangency principle for spacelike hypersurfaces with constant mean curvature and we will derive some results that might be interesting by theirself (Theorems 5 and 7).

Let u and v be two functions that are local expressions of two spacelike hypersurfaces Σ_u and Σ_v of $\mathbb{L}^{n+1}$. If Σ_u and Σ_v have a common point $p = (p_1, \ldots, p_{n+1})$ where they are tangent, we will say that Σ_u lies above Σ_v near p when $u \geq v$ on a certain neighborhood of the point $(p_1, \ldots, p_n)$. Let us assume that Σ_u and Σ_v have the same constant mean curvature H. Since Eq. (6) is of quasi-linear elliptic type, the difference function $u - v$ satisfies a *linear* elliptic equation on a neighborhood of $(p_1, \ldots, p_n)$ and the Hopf maximum principle for linear elliptic equations can be applied to $u - v$ (see [10, Th. 9.2], or [11]). Consequently, we have proved the following result.

Theorem 4 (Tangency principle). *Let Σ_1 and Σ_2 be two spacelike hypersurfaces of $\mathbb{L}^{n+1}$ with the same constant mean curvature* (*with respect to the future-directed unit normal*).

Suppose that they are tangent at a common interior point p and that Σ_1 lies above Σ_2 near p. Then they coincide in a neighborhood of p. The same holds if p is a common boundary point with the extra hypothesis that $\partial\Sigma_1$ and $\partial\Sigma_2$ are tangent at p.

This result is a stronger that the *Comparison Principle* which states that if Σ_1 and Σ_2 are two spacelike hypersurfaces with (non-necessarily constant) mean curvatures H_1 and H_2 and if and Σ_1 is locally above Σ_2 near some point p, then $H_1(p) \geq H_2(p)$.

The tangency principle allows to understand the structure of the family of compact spacelike hypersurfaces whose mean curvature function is constant. For this, let us recall that for each $H \in \mathbb{R}$, the space $\mathbb{L}^{n+1}$ can be foliated by spacelike hypersurfaces with constant mean curvature H; namely, if $H = 0$, a family of parallel spacelike hyperplanes, if H is positive, the hyperbolic hyperplanes $\{\mathcal{H}^+_{1/H}(t\mathbf{e}_{n+1}); t \in \mathbb{R}\}$ and if H is negative, by the family $\{\mathcal{H}^-_{-1/H}(t\mathbf{e}_{n+1}); t \in \mathbb{R}\}$.

As a consequence of the tangency principle, we establish two results on compact spacelike hypersurfaces. First we will consider conditions for a compact spacelike hypersurface with constant mean curvature to be contained in a halfspace determined by an umbilical hypersurface.

Theorem 5. *Let Σ be a compact spacelike hypersurface with constant mean curvature H. If the boundary $\partial\Sigma$ of Σ is included either in a hyperplane or in a hyperbolic hyperplane Π, then Σ lies completely included in one of the two halfspaces determined by Π.*

Proof. Let us assume the contrary, that is, Σ has (interior) points in both sides of Π. First we will prove the result under the assumption that $\partial\Sigma$ lies in a hyperplane Π. This hyperplane must be spacelike and we may assume, without loss of generality that $\Pi = \{x \in \mathbb{L}^{n+1}; \langle x, \mathbf{e}_{n+1}\rangle = 0\}$. Consider a family $\Pi(t) = \{x \in \mathbb{L}^{n+1}; x_{n+1} = t\}$ of parallel horizontal hyperplanes. Because Σ is a compact hypersurface and has interior points over Π, there exists $t_0 > 0$ such that

$$\Pi(t) \cap \Sigma = \emptyset \quad \text{for all } t > t_0$$

and

$$\Pi(t_0) \cap \Sigma \neq \emptyset$$

at some common interior point. Then the comparison principle implies that $H \leq 0$. Recall that all our hypersurfaces are future-directed oriented. But the tangency principle discards the case $H = 0$ because in that case, Σ would be in the hyperplane $\Pi(t_0)$ in contradiction with that $\partial\Sigma \subset \Pi(0)$. A similar argument by using hyperplanes $\Pi(t)$ with negative values for t yields $H > 0$, which is a contradiction.

We now consider that $\partial\Sigma$ lies in a hyperbolic hyperplane. After a homothety followed of an isometry of $\mathbb{L}^{n+1}$, we can assume that $\partial\Sigma$ is included in the upper hyperbolic hyperplane $\mathcal{H}^+_1(0)$. Again and by contradiction, we assume that Σ have interior points in both sides of $\mathcal{H}^+_1(0)$. Consider now the family $\mathcal{H}^+_1(t\mathbf{e}_{n+1})$ of upper hyperbolic hyperplanes, that is, vertical translations of $\mathcal{H}^+_1(0)$. Since Σ is a compact hypersurface, for a large value of t, say t_1, $\mathcal{H}^+_1(t_1\mathbf{e}_{n+1}) \cap \Sigma = \emptyset$. Now, move downwards $\mathcal{H}^+_1(t_1\mathbf{e}_{n+1})$ making t decrease from

t_1 to 0. Because there are points of Σ that lie over $\mathcal{H}_1^+(0)$, then for some $t_0 > 0$ we will have $\Sigma \cap \mathcal{H}_1^+(t_0\mathbf{e}_{n+1}) \neq \emptyset$ at some interior point and $\Sigma \cap \mathcal{H}_1^+(t\mathbf{e}_{n+1}) = \emptyset$ for $t > t_0$. The comparison principle implies that $H \leq 1$. But if $H = 1$, the tangency principle implies that $\Sigma \subset \mathcal{H}_1^+(t_0\mathbf{e}_{n+1})$, which is impossible since the points of $\partial\Sigma$ are not in $\mathcal{H}_1^+(t_0\mathbf{e}_{n+1})$. As conclusion, $H < 1$. Doing the same reasoning with $\mathcal{H}_1^+(t\mathbf{e}_{n+1})$ with $t < 0$, there exists $t_2 < 0$ such that Σ lies above $\mathcal{H}_1^+(t_2\mathbf{e}_{n+1})$ and both surfaces are tangent at some common interior point. Now the comparison principle implies that $H \geq 1$. This is a contradiction, which completes the proof of Theorem 5. □

Remark 6. The same argument leads to the following conclusions. First, if the mean curvature function of Σ does not vanish, and if its boundary is included in a hyperplane Π, then Σ lies in one of the two closed hyperspaces determined by Π. Second, and assuming that the mean curvature H is constant and the boundary lies in a hyperbolic hyperplane with the same mean curvature H, then Σ is included in this hyperbolic hyperplane.

We conclude this section by establishing, as an application of the tangency principle, a C^0-estimate of a solution of Eq. (6). As we mentioned in the Introduction, it is not possible to obtain C^0-estimates depending only on H. In the following theorem, we derive estimates of the height in terms of H and the diameter of the domain Ω.

Theorem 7. *Let Ω be a compact domain of $\mathbb{R}^n$ with diameter $\delta > 0$ and let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a function whose graph has (non-necessarily constant) mean curvature H. Let us assume that there exists $c > 0$ such that $|H| \leq c$. Then*

$$\min_{\partial\Omega} u + \frac{1}{c}\left(1 - \frac{1}{2}\sqrt{4+\delta^2c^2}\right) \leq u \leq \max_{\partial\Omega} u + \frac{1}{c}\left(-1 + \frac{1}{2}\sqrt{4+\delta^2c^2}\right). \tag{9}$$

In the particular case that $H \equiv 0$, then $\min_{\partial\Omega} u \leq u \leq \max_{\partial\Omega} u$.

Proof. We may assume, without loss of generality, that Ω is included in the disc $D_r(0)$ where $r = \delta/2$ and we set $v_\rho^\pm$ be the functions whose graphs are the hyperbolic caps $\mathcal{H}_\rho^\pm(0, r)$ by setting $\rho = 1/c$ (see notation in above section). Denote Σ the graph of u. The vertical translations of $\mathcal{H}_\rho^+(0, r)$ are the hyperbolic caps $\mathcal{H}_\rho^+(t\mathbf{e}_{n+1}, r)$. Translate $\mathcal{H}_\rho^+(0, r)$ vertically downward until it is disjoint from Σ. Then reascend $\mathcal{H}_\rho^+(0, r)$ until the first time t_0 that $\mathcal{H}_\rho^+(t_0\mathbf{e}_{n+1}, r)$ touches Σ at some point p. Then

$$v^+_{\rho|\partial D_r(0)} \leq \min_{\partial\Omega} u,$$

that is,

$$t_0 + \sqrt{\rho^2 + r^2} \leq \min_{\partial\Omega} u. \tag{10}$$

We have two possibilities about the point p. First, it is a tangent point between Σ and $\mathcal{H}_\rho^+(t_0\mathbf{e}_{n+1}, r)$ (whether p is an interior or boundary point). Because the mean curvature of $\mathcal{H}_\rho^+(t_0\mathbf{e}_{n+1}, r)$ is c and $|H| \leq c$, the tangency principle says us that $|H| \equiv c$, $\Sigma \subset \mathcal{H}_\rho^+(t_0\mathbf{e}_{n+1}, r)$ and we have equality in (10) and consequently in (9).

The other possibility is that the point p is a boundary point where both surfaces Σ and $\mathcal{H}_\rho^+(t_0\mathbf{e}_{n+1}, r)$ are not tangent. Then we have equality in (10), that is, $t_0 + \sqrt{\rho^2 + r^2} = \min_{\partial\Omega} u$ and for any $x \in \Omega$:

$$u(x) \geq v_\rho^+(x) \geq v_\rho^+(0) = t_0 + \rho = \min_{\partial\Omega} u + \rho - \sqrt{r^2 + \rho^2}.$$

So (remembering that $r = \delta/2$ and $\rho = 1/c$) we have the estimate of the left-hand side of (9). In order to prove the other inequality of (9), we use hyperbolic caps of type $\mathcal{H}_\rho^-(t\mathbf{e}_{n+1}, r)$.

For the case $H \equiv 0$, the proof can be achieved by letting $c \to 0$. This finishes the proof of Theorem 7. □

As a corollary, it may be interesting to remark the case that the mean curvature is constant and that the boundary is included in a hyperplane.

Corollary 8. *Let $\Omega \subset \mathbb{R}^n$ be a smooth domain with diameter δ. Let H be a real number and let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of (6) with the boundary condition $u = \alpha$, $\alpha \in \mathbb{R}$. Then*

$$|u - \alpha| \leq \frac{1}{|H|}\left(-1 + \frac{1}{2}\sqrt{4 + H^2\delta^2}\right)$$

and the equality holds if and only if the graph of u is a hyperbolic cap.

4. Proof of Theorem 1

We proceed to prove Theorem 1. In what follows, we return to the case $n = 2$. We begin proving that if the surface is a hyperbolic cap, we have equality in (1): consider $\mathcal{H}_\rho^-(0, r)$ with $\rho = 1/|H|$. Then $\mathcal{H}_\rho^-(0, r) \subset \{x \in \mathbb{L}^3; x_3 \geq -(1/|H|)\sqrt{1 + H^2r^2}\}$ and boundary $\partial\mathcal{H}_\rho^-(0, r) \subset \{x \in \mathbb{L}^3; x_3 = -(1/|H|)\sqrt{1 + H^2r^2}\}$. It follows that $h = (\sqrt{1 + H^2r^2} - 1)/|H|$ and

$$\text{area}(\mathcal{H}_\rho^-(0, r)) = \frac{2\pi}{H^2}\left(\sqrt{1 + H^2r^2} - 1\right).$$

Let us prove inequality (1). Consider Σ a surface in the hypothesis of Theorem 1. Let a be the unit future-directed timelike vector in $\mathbb{L}^3$ such that $\Pi = a^\perp$. We realize an isometry of the ambient space and we assume that $a = \mathbf{e}_3 = (0, 0, 1)$ and $\Pi = \{x \in \mathbb{L}^3; \langle x, \mathbf{e}_3\rangle = 0\}$. Denote $\Gamma = x(\partial\Sigma)$. If the mean curvature is $H = 0$, then $x(\Sigma)$ is the very planar domain determined by Γ. In this case, its height is $h = 0$ and we have equality in (4). Let us assume then $H \neq 0$. Consider the function $f : \Sigma \to \mathbb{R}$ given by $f = -\langle x, \mathbf{e}_3\rangle$ which measures the height of Σ with respect to the plane Π. We need to estimate the height h of de surface, that is, the number $h = \max_{p\in\Sigma} f(p) \geq 0$. As in [3], we will use the co-area formula for the function f (see [9, Th. 3.2.12]). The scheme is as follows. Denote

$$\Sigma(t) = \{p \in \Sigma; f(p) \geq t\}, \qquad \Gamma(t) = \{p \in \Sigma; f(p) = t\}.$$

We set $A(t)$ and $L(t)$ the area and the length of $\Sigma(t)$ and $\Gamma(t)$, respectively.

We compute the critical points of the function f. Because $\nabla\langle x, \mathbf{e_3}\rangle = \mathbf{e}_3 - \langle N, \mathbf{e}_3\rangle N$, $p \in \Sigma$ is a critical point of f if $|\nabla f|^2(p) = -1 + \langle N(p), \mathbf{e}_3\rangle = 0$, that is, if $N(p) = \mathbf{e}_3$. Define on Σ the functions $g_i = \langle N, \mathbf{e}_i\rangle$, $i = 1, 2$, where $\mathbf{e}_1 = (1, 0, 0)$ and $\mathbf{e}_2 = (0, 1, 0)$. Therefore, the set $\mathcal{C}$ of critical points of f is contained in $\mathcal{N}_1 \cap \mathcal{N}_2$, where

$$\mathcal{N}_1 = \{p \in \Sigma; \langle N(p), \mathbf{e}_1\rangle = 0\}, \qquad \mathcal{N}_2 = \{p \in \Sigma; \langle N(p), \mathbf{e}_2\rangle = 0\}$$

are the nodal lines of g_1 and g_2, respectively.

On the other hand, given a fixed vector $a \in \mathbb{L}^3$, the constancy of the mean curvature H of the immersion x gives the following formula:

$$\Delta\langle N, a\rangle = (4H^2 + 2K)\langle N, a\rangle = \mathrm{trace}(A^2)\langle N, a\rangle, \tag{11}$$

where Δ is the Laplacian in the induced metric by x and K the Gaussian curvature of Σ. It follows from this equation that the functions g_1 and g_2 satisfy

$$\Delta g_i - \mathrm{trace}(A^2)g_i = 0, \quad i = 1, 2. \tag{12}$$

If the functions g_i are identically zero, then $N = \mathbf{e}_3$ on Σ and the surface would be a planar domain. This yields $H = 0$, on the contrary to the assumption. Thus, either g_1 or g_2 is not trivial. Since Eq. (12) are of Schrödinger type, Cheng's theorem on nodal lines assures that the nodal line of g_i is a finite number of immersed circles [6]. In particular, its measure is zero and so $\mathcal{C}$ as well. It follows that $A(t)$ is a continuous function and the co-area formula assures

$$A'(t) = -\int_{\Gamma(t)} \frac{1}{|\nabla f|}\,\mathrm{d}s_t, \quad t \in \mathbb{R},$$

where $\mathrm{d}s_t$ is the line element on the level $\Gamma(t)$. By Hölder's inequality there holds

$$L(t)^2 = \left(\int_{\Gamma(t)} \mathrm{d}s_t\right)^2 \le \int_{\Gamma(t)} |\nabla f|\,\mathrm{d}s_t \int_{\Gamma(t)} \frac{1}{|\nabla f|}\,\mathrm{d}s_t = -A'(t)\int_{\Gamma(t)} |\nabla f|\,\mathrm{d}s_t. \tag{13}$$

Recall that $|\nabla f|$ along the curve $\Gamma(t)$ is

$$|\nabla f|^2 = -1 + \langle N, \mathbf{e}_3\rangle^2 = \langle \nu_t, \mathbf{e}_3\rangle^2,$$

where ν_t is the unit inner conormal of $\Sigma(t)$ along $\Gamma(t) = \partial\Sigma(t)$. As $\Sigma(t)$ is above the plane $\Pi(t) = \{x \in \mathbb{L}^3; -\langle x, \mathbf{e}_3\rangle = t\}$, we know $\langle \nu_t, \mathbf{e}_3\rangle \le 0$. Hence

$$|\nabla f|_{|\Gamma(t)} = -\langle \nu_t, \mathbf{e}_3\rangle.$$

It follows from (13) that

$$L(t)^2 \le A'(t)\int_{\Gamma(t)} \langle \nu_t, \mathbf{e}_3\rangle\,\mathrm{d}s_t, \quad t \in \mathbb{R}. \tag{14}$$

We know that $\Sigma(t)$ is a compact surface with smooth boundary $\Gamma(t)$ for almost $t \in \mathbb{R}$. If $t \ge 0$, $\Gamma(t) \subset \Pi(t)$ and by the flux formula (8), we have

$$-\int_{\partial\Sigma(t)} \langle \nu_t, \mathbf{e}_3\rangle\,\mathrm{d}s_t = 2|H||a_g(t)|,$$

where $a_g(t)$ is the algebraic area of the planar closed curve $\Gamma(t)$. Thus (14) can be written as

$$L(t)^2 \le -2|H|A'(t)|a_g(t)|. \tag{15}$$

If $t < 0$, then $\partial\Sigma \subset \Sigma(t)$ and so $\Gamma(t)$ has a component in the plane Π and possibly others in $\Pi(t)$.

Denote by $\Omega_1(t), \ldots, \Omega_{n_t}(t)$ the bounded domains determined by $\Pi(t) \cap \Sigma(t)$ and let $a_i(t)$ be the Lebesgue area of the corresponding $\Omega_i(t)$. Then

$$a_g(t) = \epsilon_1 a_1(t) + \cdots + \epsilon_{n_t} a_{n_t}(t),$$

where $\epsilon_i \in \mathbb{Z}$ is the order number corresponding to the curves $x(\partial\Omega_i(t))$. Then

$$|a_g(t)| \le |\epsilon_1| a_1(t) + \cdots |\epsilon_{n_t}| a_{n_t}(t). \tag{16}$$

If $L_i(t)$ denotes the length of $\partial\Omega_i(t)$ we have

$$L(t) = |\epsilon_1| L_1(t) + \cdots + |\epsilon_{n_t}| L_{n_t}(t),$$

which implies that

$$\epsilon_1^2 L_1(t)^2 + \cdots + \epsilon_{n_t}^2 L_{n_t}(t)^2 \le L(t)^2. \tag{17}$$

By virtue of inequalities (15)–(17), we have

$$\epsilon_1^2 L_1(t)^2 + \cdots + \epsilon_{n_t}^2 L_{n_t}(t)^2 \le -2|H|A'(t)(|\epsilon_1| a_1(t) + \cdots + |\epsilon_{n_t}| a_{n_t}(t)).$$

We use the isoperimetric inequality in the plane $\Pi(t)$. We note that $\Pi(t)$ is isometric to the Euclidean plane $\mathbb{R}^2$, and then, such inequality holds for such planes. Therefore $4\pi a_i(t) \le L_i(t)^2$ and if we take into account that $|\epsilon_i| \le \epsilon_i^2$, we have

$$2\pi \le -|H|A'(t) \quad \text{for every } t \ge 0. \tag{18}$$

Integrating this inequality from 0 to the height h, we obtain

$$2\pi h \le |H|(A(0) - A(h)) = |H|A,$$

which yields the desired estimate (1).

To finish the proof we analyze the equality in (1). In such case, we have also equality in (13) and in the isoperimetric inequality. Thus $|\nabla f|$ is constant function in each $\Gamma(t)$ and $\Gamma(t)$ is a circle, for each $t > 0$. However, the only compact spacelike surfaces of $\mathbb{L}^3$ with non-zero constant mean curvature and bounded by a circle are hyperbolic caps, see [1]. This proves that $x(\Sigma)$ is a hyperbolic cap and we conclude the proof of Theorem 1.

An immediate consequence of Theorem 1 is the following corollary.

Corollary 9. *Let $x : \Sigma \to \mathbb{L}^3$ be an immersion as in Theorem 1 except that the boundary of the surface is not planar. Denote x_3 the height function on Σ given by $x_3(p) = -\langle p, \mathbf{e}_3\rangle$. Then*

$$\min_{\partial\Sigma} x_3 - \frac{\text{area}(\Sigma)|H|}{2\pi} \le x_3 \le \max_{\partial\Sigma} x_3 + \frac{\text{area}(\Sigma)|H|}{2\pi}.$$

Remark 10. We give the following interpretation of Theorem 1: any compact spacelike surface Σ of $\mathbb{L}^3$ with constant mean curvature and with planar boundary has area bigger than the area of a hyperbolic cap with the same height and mean curvature than Σ.

Remark 11. It remains as an open problem the extension of the height estimate (1) for arbitrary dimension. We conjecture that for compact spacelike hypersurfaces Σ of $\mathbb{L}^{n+1}$ with constant mean curvature H and whose boundary lies in a hyperplane, it holds the following estimate of the height H of Σ:

$$h \leq \frac{\mathrm{vol}(\Sigma)|H|}{\omega_{n-1}},$$

where ω_{n-1} is the volume of the $(n-1)$-dimensional unit sphere.

Acknowledgements

The author is partially supported by MCYT-FEDER Grant no. BFM2001-2967.

References

[1] L. Alías, R. López, J. Pastor, Compact spacelike surfaces with constant mean curvature in the Lorentz–Minkowski 3-space, Tohoku Math. J. 50 (1998) 491–501.
[2] L. Alías, J. Pastor, Constant mean curvature spacelike hypersurfaces with spherical boundary in the Lorentz–Minkowski space, J. Geom. Phys. 28 (1998) 85–93.
[3] R. Bartnik, Maximal surfaces and general relativity, in: Proceedings of the Miniconference on Geometry and Partial Differential Equations, vol. 2, Canberra, 1986;
R. Bartnik, Maximal surfaces and general relativity, in: Proceedings of the Centre of Mathematical Analysis, vol. 12, Australian National University, Canberra, 1987, pp. 24–49.
[4] R. Bartnik, L. Simon, Spacelike hypersurfaces with prescribed boundary values and mean curvature, Commun. Math. Phys. 87 (1982) 131–152.
[5] H. Bray, Black holes, geometric flows, and the Penrose inequality in general relativity, Notices AMS 49 (2002) 1372–1381.
[6] S.Y. Cheng, Eigenfunctions and nodal sets, Comm. Math. Helv. 51 (1976) 43–55.
[7] Y. Choquet-Bruhat, J. York, The cauchy problem, in: A. Held. (Ed.), General Relativity and Gravitation, Plenum Press, New York, 1980.
[8] T.H. Colding, W.P. Minicozzi II, Minimal Surfaces, Lecture Notes, vol. 4, Courant Institute of Mathematical Sciences, 1999.
[9] H. Federer, Geometric Measure Theory, Springer-Verlag, Berlin, 1969.
[10] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, 1983.
[11] E. Hopf, Elementare Bermekungen Über die Lösungen partieller differentialgleichunger zweiter ordnung von elliptischen typen, Preuss. Akad. Wiss. 19 (1927) 147–152.
[12] J.E. Marsden, F.J. Tipler, Maximal hypersurfaces and foliations of constant mean curvature in general relativity, Phys. Rep. 66 (1980) 109–139.
[13] L. Simon, Lectures on Geometric Measure Theory, Centre for Mathematical Analysis, Australian Natioanl University, 1984.
[14] A. Treibergs, Entire spacelike hypersurfaces of constant mean curvature in Minkowski space, Invent. Math. 66 (1982) 39–56.